Buckling and Vibration of Laminated Composite Plates Using Various Plate Theories

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Analytical and finite-element solutions of the classical, first-order, and third-order laminate theories are developed to study the buckling and free-vibration behavior of cross-ply rectangular composite laminates under various boundary conditions. The effects of side-to-thickness ratio, aspect ratio, and lamination schemes on the fundamental frequencies and critical buckling loads are investigated. The study concludes that shear deformation laminate theories accurately predict the behavior of composite laminates, whereas the classical laminate theory overpredicts natural frequencies and buckling loads.

I. Introduction

THE analyses of laminated composite plates are often based on equivalent single-layer theories, commonly referred to as laminate plate theories. These theories are derived from the three-dimensional elasticity equations by approximating the thickness variation of the displacement field. In the classical laminate plate theory (CPT) it is assumed that (the Kirchhoff hypothesis) 1) the straight lines do not undergo axial deformation (i.e., inextensible); 2) straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation, and 3) the straight lines rotate such that they remain perpendicular to the midsurface after deformation.

The first two assumptions imply that the transverse displacement is independent of the transverse (or thickness) coordinate and the transverse normal strain is zero. The third assumption results in zero transverse shear strains. Thus, in the classical plate theory all transverse stresses are neglected.

Shear deformation theories are those in which the transverse shear stresses are accounted for. Such theories can be used to describe the kinematics of deformation of laminated plates accurately. The first-order shear-deformation theory (FSDT) (see Reddy[1-3] and Reddy and Chao[4]), commonly known as the Mindlin plate theory, accounts for layerwise constant states of transverse shear stresses (i.e., assumption 3 is removed), whereas the higher-order shear-deformation theory (HSDT) advanced by Reddy[1,5,6] and Reddy and Phan[7] accounts for layerwise parabolic distribution of transverse shear stresses (i.e., assumptions 2 and 3 are removed). The exact solution of laminated plate theories is often limited to rectangular geometries, simply supported boundary conditions, and linear theories (see Refs. 1, 4–12). Finite-element analysis of the classical theory, and first-order and third-order shear deformation laminate theories for arbitrary loading, geometry, and boundary conditions, have been known (see Refs. 13–19). Recently, the authors have developed the Lévy-type solutions to various laminate theories applied to rectangular laminates under various types of boundary conditions.[20] This study dealt with the bending of composite laminates, and analytical solutions for buckling and vibration for various boundary conditions were not reported there.

In the present study, exact and finite-element solutions for the free vibration and buckling of cross-ply rectangular composite laminates are developed using the classical, first-order, and third-order laminate plate theories under various boundary conditions. A comparison of the fundamental frequencies and critical buckling loads predicted by the three theories is presented.

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II. Governing Equations

A. Kinematic Relations

The third-order shear deformation theory used in the present study is based on the following representation of the displacement field across the plate thickness[5,6]:

$$u_1(x,y,z,t) = u + z\left[\phi_1 - \alpha\frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_1 + \frac{\partial w}{\partial x}\right)\right] \tag{1a}$$

$$u_2(x,y,z,t) = v + z\left[\phi_2 - \alpha\frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_2 + \frac{\partial w}{\partial y}\right)\right] \tag{1b}$$

$$u_3(x,y,z,t) = w \tag{1c}$$

Here u, v, and w denote the displacement components in the x, y, and z directions, respectively, at time t; and ϕ_1 and ϕ_2 are the rotations of the transverse normals about the y and x axes, respectively. All of the generalized displacements (u,v,w,ϕ_1,ϕ_2) are functions of x, y, and t. Note that the displacement field of the first-order shear deformation theory can be deduced from Eq. (1) by setting $\alpha = 0$. The displacement field of the classical laminate theory can be obtained from that of the first-order theory by setting

$$\phi_1 = -\frac{\partial w}{\partial x}, \qquad \phi_2 = -\frac{\partial w}{\partial y} \tag{2}$$

B. Equations of Motion

The equations of motion appropriate for the displacement field, Eq. (1), can be derived using the dynamic version of the principle of virtual displacements[1]:

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_1\ddot{u} + \bar{I}_2\ddot{\phi}_1 - \alpha c_2 I_4\frac{\partial \ddot{w}}{\partial x} \tag{3a}$$

$$\frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = I_1\ddot{v} + \bar{I}_2\ddot{\phi}_2 - \alpha c_2 I_4\frac{\partial \ddot{w}}{\partial y} \tag{3b}$$

$$\frac{\partial \hat{Q}_1}{\partial x} + \frac{\partial \hat{Q}_2}{\partial y} + \bar{N}_1\frac{\partial^2 w}{\partial x^2} + \bar{N}_2\frac{\partial^2 w}{\partial y^2} + \bar{N}_6\frac{\partial^2 w}{\partial x\,\partial y} + q + \alpha c_2\left(\frac{\partial^2 P_1}{\partial x^2} + 2\frac{\partial^2 P_6}{\partial x\,\partial y} + \frac{\partial^2 P_2}{\partial y^2}\right)$$
$$= I_1\ddot{w} - \alpha c_2^2 I_7\left(\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\partial^2\ddot{w}}{\partial y^2}\right) + \alpha c_2\left[I_4\left(\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\ddot{v}}{\partial y}\right) + \bar{I}_5\left(\frac{\partial\ddot{\phi}_1}{\partial x} + \frac{\partial\ddot{\phi}_2}{\partial y}\right)\right] \tag{3c}$$

$$\frac{\partial \hat{M}_1}{\partial x} + \frac{\partial \hat{M}_6}{\partial y} - \hat{Q}_1 = \bar{I}_2 \ddot{u} + \bar{I}_3 \ddot{\phi}_1 - \alpha c_2 \bar{I}_5 \frac{\partial \ddot{w}}{\partial x} \quad (3d)$$

$$\frac{\partial \hat{M}_6}{\partial x} + \frac{\partial \hat{M}_2}{\partial y} - \hat{Q}_2 = \bar{I}_2 \ddot{v} + \bar{I}_3 \ddot{\phi}_2 - \alpha c_2 \bar{I}_5 \frac{\partial \ddot{w}}{\partial y} \quad (3e)$$

where $c_1 = 4/h^2$, $c_2 = c_1/3$, and

$$\hat{M}_i = M_i - \alpha c_2 P_i, \qquad \hat{Q}_i = Q_i - \alpha c_1 R_i \quad (4)$$

and the superposed dot denotes differentiation with respect to time, q is the distributed transverse load, and (N_i, M_i, P_i) are the stress resultants

$$(N_i, M_i, P_i) = \sum_{m=1}^{L} \int_{z_m}^{z_{m+1}} \sigma_i^{(m)}(1, z, z^3)\,dz, \qquad i = 1,2,6 \quad (5a)$$

$$(Q_1, R_1) = \sum_{m=1}^{L} \int_{z_m}^{z_{m+1}} \sigma_5^{(m)}(1, z^2)\,dz \quad (5b)$$

$$(Q_2, R_2) = \sum_{m=1}^{L} \int_{z_m}^{z_{m+1}} \sigma_4^{(m)}(1, z^2)\,dz \quad (5c)$$

Here $\bar{N}_1$, $\bar{N}_2$, and $\bar{N}_6$ are the constant in-plane edge loads, and L denotes the total number of layers in the laminate. The inertias I_i $(i = 1,2,3,4,5,7)$ are defined by

$$(I_1, I_2, I_3, I_4, I_5, I_7) = \sum_{m=1}^{L} \int_{z_m}^{z_{m+1}} \rho^{(m)}(1, z, z^2, z^3, z^4, z^6)\,dz \quad (6a)$$

where $\rho^{(m)}$ is the material density of the mth layer,

$$\bar{I}_2 = I_2 - \alpha c_2 I_4, \qquad \bar{I}_5 = I_5 - \alpha c_2 I_7$$

$$\bar{I}_3 = I_3 - 2\alpha c_2 I_5 + \alpha c_2^2 I_7 \quad (6b)$$

The force and moment resultants can be expressed in terms of the displacements using the laminate constitutive equations[1]:

$$N_i = A_{i1}\frac{\partial u}{\partial x} + A_{i2}\frac{\partial v}{\partial y} + A_{i6}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \hat{B}_{i1}\frac{\partial \phi_1}{\partial x} + \hat{B}_{i2}\frac{\partial \phi_2}{\partial y} + \hat{B}_{i6}\left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}\right) - \alpha c_2\left(E_{i1}\frac{\partial^2 w}{\partial x^2} + E_{i2}\frac{\partial^2 w}{\partial y^2} + 2E_{i6}\frac{\partial^2 w}{\partial x\,\partial y}\right) \quad (7a)$$

$$M_i = B_{i1}\frac{\partial u}{\partial x} + B_{i2}\frac{\partial v}{\partial y} + B_{i6}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \hat{D}_{i1}\frac{\partial \phi_1}{\partial x} + \hat{D}_{i2}\frac{\partial \phi_2}{\partial y} + \hat{D}_{i6}\left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}\right) - \alpha c_2\left(F_{i1}\frac{\partial^2 w}{\partial x^2} + F_{i2}\frac{\partial^2 w}{\partial y^2} + 2F_{i6}\frac{\partial^2 w}{\partial x\,\partial y}\right) \quad (7b)$$

$$P_i = E_{i1}\frac{\partial u}{\partial x} + E_{i2}\frac{\partial v}{\partial y} + E_{i6}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \hat{F}_{i1}\frac{\partial \phi_1}{\partial x} + \hat{F}_{i2}\frac{\partial \phi_2}{\partial y} + \hat{F}_{i6}\left(\frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}\right) - \alpha c_2\left(H_{i1}\frac{\partial^2 w}{\partial x^2} + H_{i2}\frac{\partial^2 w}{\partial x^2} + 2H_{i6}\frac{\partial^2 w}{\partial x\,\partial y}\right) \quad (7c)$$

$$Q_1 = \hat{A}_{55}\left(\phi_1 + \frac{\partial w}{\partial x}\right) + \hat{A}_{45}\left(\phi_2 + \frac{\partial w}{\partial y}\right) \quad (7d)$$

$$Q_2 = \hat{A}_{45}\left(\phi_1 + \frac{\partial w}{\partial x}\right) + \hat{A}_{44}\left(\phi_2 + \frac{\partial w}{\partial y}\right) \quad (7e)$$

$$R_1 = \hat{D}_{55}\left(\phi_1 + \frac{\partial w}{\partial x}\right) + \hat{D}_{45}\left(\phi_2 + \frac{\partial w}{\partial y}\right) \quad (7f)$$

$$R_2 = \hat{D}_{45}\left(\phi_1 + \frac{\partial w}{\partial x}\right) + \hat{D}_{44}\left(\phi_2 + \frac{\partial w}{\partial y}\right) \quad (7g)$$

where

$$\hat{B}_{ij} = B_{ij} - \alpha c_2 E_{ij}, \qquad \hat{D}_{ij} = D_{ij} - \alpha c_2 F_{ij},$$

$$\hat{F}_{ij} = F_{ij} - \alpha c_2 H_{ij}, \qquad i,j = 1,2,6 \quad (8a)$$

$$\hat{A}_{ij} = A_{ij} - \alpha c_1 D_{ij}, \qquad \hat{D}_{ij} = D_{ij} - \alpha c_1 F_{ij}, \qquad i,j = 4,5 \quad (8b)$$

and $A_{ij}, B_{ij}, \ldots$ are the laminate stiffnesses.

For antisymmetric cross-ply laminates, the following laminate stiffnesses are identically zero:

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = A_{45} = 0 \quad (9a)$$

$$E_{16} = E_{26} = F_{16} = F_{26} = H_{16} = H_{26} = D_{45} = F_{45} = 0 \quad (9b)$$

III. The Lévy-Type Solution

A. Solution Procedure

A generalized Lévy-type solution, in conjunction with the state-space concept, is used to analyze the free-vibration and buckling problems of antisymmetric cross-ply laminated rectangular plates. The edges ($y = 0$ and $y = b$) are assumed to be simply supported, while the remaining ones ($x = \pm a/2$) may have arbitrary combinations of free, clamped, and simply supported edge conditions. We express the generalized displacements as products of undetermined functions and known trigonometric functions so as to satisfy identically the simply supported boundary conditions at $y = 0$ and $y = b$:

$$u = w = \phi_1 = N_2 = M_2 = P_2 = 0, \qquad \text{for HSDT} \quad (10a)$$

$$u = w = \phi_1 = N_2 = M_2 = 0, \qquad \text{for FSDT} \quad (10b)$$

$$u = w = N_2 = M_2 = 0, \qquad \text{for CPT} \quad (10c)$$

For the free-vibration case, we set the load terms $\bar{N}_1$, $\bar{N}_2$, and $\bar{N}_6$, and q in the governing equations to zero, and represent the displacement quantities as

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \phi_1(x,y,t) \\ \phi_2(x,y,t) \end{Bmatrix} = \begin{Bmatrix} U_m(x)\sin\beta y \\ V_m(x)\cos\beta y \\ W_m(x)\sin\beta y \\ X_m(x)\sin\beta y \\ Y_m(x)\cos\beta y \end{Bmatrix} e^{i\omega_m t} \quad (11)$$

Here $\beta \equiv m\pi/b$ and ω_m denotes the eigenfrequency associated with the mth eigenmode. The representation (11) is valid for HSDT, FSDT, and CPT.

Substitution of Eq. (11) into Eqs. (7) and the result into Eqs. (3), we obtain five differential equations for HSDT and FSDT and three differential equations for CPT. In order to represent the system of differential equations in the form needed for the state-space solution procedure, the following variables are introduced:

HSDT:

$$Z_1 = U_m, \quad Z_2 = U'_m, \quad Z_3 = V_m, \quad Z_4 = V'_m,$$
$$Z_5 = W_m, \quad Z_6 = W'_m, \quad Z_7 = W''_m,$$
$$Z_8 = W'''_m, \quad Z_9 = X_m, \quad Z_{10} = X'_m,$$
$$Z_{11} = Y_m, \quad Z_{12} = Y'_m \quad (12)$$

FSDT:

$$Z_1 = U_m, \quad Z_2 = U'_m, \quad Z_3 = V_m, \quad Z_4 = V'_m,$$
$$Z_5 = W_m, \quad Z_6 = W'_m, \quad Z_7 = X_m$$
$$Z_8 = X'_m, \quad Z_9 = Y_m, \quad Z_{10} = Y'_m \quad (13)$$

CPT:

$$Z_1 = U_m, \quad Z_2 = U'_m, \quad Z_3 = V_m, \quad Z_4 = V'_m,$$
$$Z_5 = W_m, \quad Z_6 = W'_m$$
$$Z_7 = W''_m, \quad Z_8 = W'''_m \tag{14}$$

where primes over the variables indicate differentiation with respect to x. The differential equations take the form

$$\{Z'\} = [A]\{Z\} \tag{15}$$

where the matrix $[A]$ is defined in Appendix A for HSDT, FSDT, and CPT as (12×12), (10×10), and (8×8) matrices, respectively.

A formal solution to Eq. (15) is given by[21,22]

$$\{Z(x)\} = e^{Ax}\{k\} \tag{16}$$

where $\{k\}$ is a constant column vector associated with the boundary conditions and e^{Ax} is given by

$$e^{Ax} = [S]\begin{bmatrix} e^{\lambda_1 x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n x} \end{bmatrix}[S]^{-1} \tag{17}$$

The value of n is 12 for HSDT, 10 for FSDT, and 8 for CPT. Here λ_i denotes the distinct eigenvalues of $[A]$, whereas $[S]$ denotes the matrix of eigenvectors of $[A]$.

Substitution of Eq. (16) into the boundary conditions associated with the remaining two opposite edges $x = \pm a/2$ results in a homogeneous system of equations given by

$$\sum_{j=1}^{n} M_{ij} k_j = 0 \tag{18}$$

where $i = 1,12$ for HSDT, $i = 1,10$ for FSDT, and $i = 1,8$ for CPT. For the nontrivial solution of Eq. (18), the determinant should be zero:

$$|M_{ij}| = 0 \tag{19}$$

Equation (19) gives the eigenfrequencies or the buckling loads.

B. Boundary Conditions

The boundary conditions for simply supported (S), clamped (C), and free (F) at the edges $x = \pm a/2$ for the three theories are

HSDT:

$$\text{S:} \quad v = w = \phi_2 = N_1 = M_1 = P_1 = 0$$

$$\text{C:} \quad u = v = w = \frac{\partial w}{\partial x} = \phi_1 = \phi_2 = 0$$

$$\text{F:} \quad N_1 = N_6 = M_1 = P_1 = \hat{M}_6 = 0$$

$$\hat{Q}_1 + c_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y}\right) = 0 \tag{20}$$

FSDT:

$$\text{S:} \quad v = w = \phi_2 = N_1 = M_1 = 0$$

$$\text{C:} \quad u = v = w = \phi_1 = \phi_2 = 0$$

$$\text{F:} \quad N_1 = M_1 = Q_1 = N_6 = M_6 = 0 \tag{21}$$

CPT:

$$\text{S:} \quad v = w = N_1 = M_1 = 0$$

$$\text{C:} \quad u = v = w = \frac{\partial w}{\partial x} = 0$$

$$\text{F:} \quad N_1 = M_1 = N_6 = 0$$

$$\frac{\partial M_1}{\partial x} + 2\frac{\partial M_6}{\partial y} = 0 \tag{22}$$

IV. The Finite-Element Formulation

This section deals with the development of finite-element models of the laminate plate theories. The finite-element model based on the total potential energy principle for the classical laminate theory and the third-order theory requires Hermite cubic or higher-order interpolation functions in the approximation of the transverse deflection. On the other hand, the finite-element model based on the total potential energy principle of the first-order shear deformation theory allows us to use linear or higher-order Lagrange interpolation functions for all displacements. Both of these models are called *displacement models.*

Let (u,v,w,ϕ_1,ϕ_2) be interpolated by expressions of the form

$$u = \sum_{j=1}^{n} u_j \psi_j(x,y), \qquad v = \sum_{j=1}^{n} v_j \psi_j(x,y)$$

$$\phi_1 = \sum_{j=1}^{n} \phi_j^1 \psi_j(x,y), \qquad \phi_2 = \sum_{j=1}^{n} \phi_j^2 \psi_j(x,y)$$

$$w = \sum_{j=1}^{m} \Delta_j \hat{\phi}_j(x,y) \tag{23}$$

Here $(u_j, v_j, \phi_j^1, \phi_j^2)$ denote the nodal values of (u,v,ϕ_1,ϕ_2) and Δ_j denote the nodal values of w and its derivatives. For linear Lagrange interpolation of (u,v,ϕ_1,ϕ_2) and Hermite cubic interpolation of w using four-node rectangular elements, we have $n = 4$ and $m = 16$[23,24]. In this case, the four nodal values associated with w are

$$w, \quad \frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \frac{\partial^2 w}{\partial x\,\partial y}$$

The element is called a C^1 element, and it has a total of eight degrees of freedom per node for the third-order theory and six degrees of freedom per node for the classical theory. For the first-order shear deformation theory ($\alpha = 0$), the number of degrees of freedom per node is five.

Substituting Eq. (23) into Eq. (3), we obtain the element equation

$$\sum_{\beta=1}^{5} \sum_{j=1}^{n(\beta)} (K_{ij}^{\alpha\beta} \Delta_j^\beta - \omega^2 M_{ij}^{\alpha\beta} \Delta_j^\beta) = 0, \qquad i = 1,2,\ldots,n(\alpha) \tag{24}$$

where $\alpha = 1,2,3$; $n(1) = n(2) = n(4) = n(5) = 4$, and $n(3) = 16$. The variables Δ_j^β, stiffness coefficients $K_{ij}^{\alpha\beta}$ (symmetric), and mass coefficients $M_{ij}^{\alpha\beta}$ (symmetric) are defined by

$$\Delta_j^1 = u_j, \quad \Delta_j^2 = v_j, \quad \Delta_j^3 = \Delta_j, \quad \Delta_j^4 = \phi_j^1, \quad \Delta_j^5 = \phi_j^2 \tag{25}$$

and

$$K_{ij}^{1\alpha} = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{1j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{6j}^\alpha\right) \mathrm{d}x\,\mathrm{d}y$$

$$K_{ij}^{2\alpha} = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} N_{6j}^\alpha + \frac{\partial \psi_i}{\partial y} N_{2j}^\alpha\right) \mathrm{d}x\,\mathrm{d}y$$

Table 1 Effect of degree of orthotropy of the individual layers on the dimensionless fundamental frequency of simply supported antisymmetric square laminates: $a/h = 5$, $\bar{\omega} = \omega(\rho h^2/E_2)^{1/2}$

Source	Number of layers	E_1/E_2: 3	10	20	30	40
Noor[34a]		0.25031	0.27938	0.30698	0.32705	0.34250
HSDT[b]		0.24877	0.27966	0.31297	0.34034	0.36362
HSDT[c]		0.24868	0.27955	0.31284	0.34020	0.36348
HSDT[d]	2	0.24868	0.27955	0.31284	0.34020	0.36348
FSDT[c]		0.24834	0.27757	0.30824	0.33284	0.35333
FSDT[d]		0.24834	0.27757	0.30824	0.33284	0.35333
CPT[c]		0.27082	0.30968	0.35422	0.39335	0.42884
CPT[d]		0.27082	0.30968	0.35422	0.39335	0.42884
Noor[a]		0.26182	0.32578	0.37622	0.40660	0.42719
HSDT[b]		0.26012	0.32791	0.38515	0.42148	0.44694
HSDT[c]		0.26003	0.32782	0.38506	0.42139	0.44686
HSDT[d]	4	0.26003	0.32782	0.38506	0.42139	0.44686
FSDT[c]		0.26017	0.32898	0.38754	0.42479	0.45083
FSDT[d]		0.26017	0.32898	0.38754	0.42479	0.45083
CPT[c]		0.28676	0.38877	0.49907	0.58900	0.66690
CPT[d]		0.28676	0.38877	0.49907	0.58900	0.66690
Noor[a]		0.26440	0.33657	0.39359	0.42783	0.45091
HSDT[b]		0.26231	0.33630	0.39681	0.43427	0.46012
HSDT[c]		0.26223	0.33621	0.39672	0.43419	0.46005
HSDT[d]	6	0.26223	0.33621	0.39672	0.43419	0.46005
FSDT[c]		0.26228	0.33673	0.39771	0.43531	0.46105
FSDT[d]		0.26228	0.33673	0.39771	0.43531	0.46105
CPT[c]		0.28966	0.40215	0.52234	0.61963	0.70359
CPT[d]		0.28966	0.40215	0.52234	0.61963	0.70359
Noor[a]		0.26583	0.34250	0.40337	0.44011	0.46498
HSDT[b]		0.26345	0.34059	0.40278	0.44086	0.46699
HSDT[c]		0.26337	0.34050	0.40270	0.44079	0.46692
HSDT[d]	10	0.26337	0.34050	0.40270	0.44079	0.46692
FSDT[c]		0.26335	0.34053	0.40255	0.44023	0.46577
FSDT[d]		0.26335	0.34053	0.40255	0.44023	0.46577
CPT[c]		0.29115	0.40888	0.53397	0.63489	0.72184
CPT[d]		0.29115	0.40888	0.53397	0.63489	0.72184

[a]Results obtained by applying a finite-difference scheme to the equations of the three-dimensional elasticity theory. [b]Results obtained using the finite-element solution.[19] [c]Results reported in Ref. 19 using the Navier solution. [d]Results obtained with the exact solution developed in this paper.

Table 2 Effect of degree of orthotropy of the individual layers on the dimensionless critical buckling loads of simply supported antisymmetric square laminates: $\bar{N}_1/\bar{N}_1 b^2/(E_2 h^3), \bar{N}^2 = 0$, $a/h = 10$

Source	Number of layers	E_1/E_2: 3	10	20	30	40
Noor[35a]		4.6948	6.1181	7.8196	9.3746	10.817
HSDT[b]		4.7769	6.2756	8.1198	9.8751	11.569
HSDT[c]		4.7749	6.2721	8.1151	9.8695	11.563
HSDT[d]	2	4.7749	6.2721	8.1151	9.8695	11.563
FSDT[c]		4.7718	6.2465	8.0423	9.7347	11.353
FSDT[d]		4.7718	6.2465	8.0423	9.7347	11.353
CPT[c]		5.0338	6.7033	8.8158	10.891	12.957
CPT[d]		5.0338	6.7033	8.8158	10.891	12.957
Noor[a]		5.1738	9.0164	13.743	17.783	21.280
HSDT[b]		5.254	9.2344	14.258	18.671	22.582
HSDT[c]		5.2523	9.2315	14.254	18.667	22.579
HSDT[d]	4	5.2523	9.2315	14.254	18.667	22.579
FSDT[c]		5.2543	9.2552	14.332	18.815	22.806
FSDT[d]		5.2543	9.2552	14.332	18.815	22.806
CPT[c]		5.5738	10.295	16.988	23.675	30.359
CPT[d]		5.5738	10.295	16.988	23.675	30.359
Noor[a]		5.267	9.6051	15.001	19.6394	23.669
HSDT[b]		5.344	9.7788	15.355	20.2038	24.462
HSDT[c]		5.342	9.7762	15.352	20.201	24.460
HSDT[d]	6	5.342	9.7762	15.352	20.201	24.460
FSDT[c]		5.343	9.7893	15.394	20.280	24.577
FSDT[d]		5.343	9.7893	15.394	20.280	24.577
CPT[c]		5.674	10.960	18.502	26.042	33.582
CPT[d]		5.674	10.960	18.502	26.042	33.582
Noor[a]		5.3159	9.9134	15.669	20.6347	24.964
HSDT[b]		5.3899	10.058	15.917	20.9887	24.424
HSDT[c]		5.3882	10.056	15.914	20.986	25.422
HSDT[d]	10	5.3882	10.056	15.914	20.986	25.422
FSDT[c]		5.3884	10.060	15.927	21.008	25.450
FSDT[d]		5.3884	10.060	15.927	21.008	25.450
CPT[c]		5.725	11.300	19.277	27.254	35.232
CPT[d]		5.725	11.300	19.277	27.254	35.232

[a]Results obtained by applying a finite-difference scheme to the equations of the three-dimensional elasticity theory. [b]Results obtained using the finite-element solution.[19] [c]Results reported in Ref. 19 using the Navier solution. [d]Results obtained with the exact solution developed in this paper.

$$K_{ij}^{3\alpha} = \int_{\Omega^e} \left[\frac{\partial \hat{\phi}_i}{\partial x} \hat{Q}_{1j}^{\alpha} + \frac{\partial \hat{\phi}_i}{\partial y} \hat{Q}_{2j}^{\alpha} - c_2 \left(\frac{\partial^2 \hat{\phi}_i}{\partial x^2} P_{1j}^{\alpha} + 2 \frac{\partial^2 \hat{\phi}_i}{\partial x \, \partial y} P_{6j}^{\alpha} + \frac{\partial^2 \hat{\phi}_i}{\partial y^2} P_{2j}^{\alpha} \right) \right] \mathrm{d}x \, \mathrm{d}y$$

$$K_{ij}^{4\alpha} = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} \hat{M}_{1j}^{\alpha} + \frac{\partial \psi_i}{\partial y} \hat{M}_{6j}^{\alpha} + \psi_i \hat{Q}_{1j}^{\alpha} \right) \mathrm{d}x \, \mathrm{d}y$$

$$K_{ij}^{5\alpha} = \int_{\Omega^e} \left(\frac{\partial \psi_i}{\partial x} \hat{M}_{6j}^{\alpha} + \frac{\partial \psi_i}{\partial y} \hat{M}_{2j}^{\alpha} + \psi_i \hat{Q}_{2j}^{\alpha} \right) \mathrm{d}x \, \mathrm{d}y$$

$$M_{ij}^{11} = I_1 S_{ij}^0, \qquad M_{ij}^{12} = 0, \qquad M_{ij}^{13} = -\alpha c_2 I_4 S_{ij}^{0x}$$

$$M_{ij}^{14} = I_2 S_{ij}^0, \qquad M_{ij}^{15} = 0, \qquad M_{ij}^{22} = I_1 S_{ij}^0,$$

$$M_{ij}^{23} = -\alpha c_2 I_4 S_{ij}^{0y}, \qquad M_{ij}^{24} = 0, \qquad M_{ij}^{25} = I_2 S_{ij}^0$$

$$M_{ij}^{33} = I_1 S_{ij}^1 + \alpha c_2^2 I_7 (S_{ij}^{xx} + S_{ij}^{yy})$$

$$M_{ij}^{34} = \bar{I}_5 S_{ij}^{0x}, \qquad M_{ij}^{35} = \bar{I}_5 S_{ij}^{0y}$$

$$M_{ij}^{44} = \bar{I}_3 S_{ij}^0, \qquad M_{ij}^{45} = 0, \qquad M_{ij}^{55} = \bar{I}_3 S_{ij}^0$$

where

$$S_{ij}^0 = \int_{\Omega^e} \psi_i \psi_j \, \mathrm{d}x \, \mathrm{d}y, \qquad S_{ij}^{0x} = \int_{\Omega^e} \psi_i \frac{\partial \hat{\phi}_j}{\partial x} \mathrm{d}x \, \mathrm{d}y$$

$$S_{ij}^{0y} = \int_{\Omega^e} \psi^i \frac{\partial \hat{\phi}_j}{\partial y} \mathrm{d}x \, \mathrm{d}y, \qquad S_{ij}^1 = \int_{\Omega^e} \hat{\phi}_i \hat{\phi}_j \, \mathrm{d}x \, \mathrm{d}y$$

$$S_{ij}^{xx} = \int_{\Omega^e} \frac{\partial \hat{\phi}_i}{\partial x} \frac{\partial \hat{\phi}_j}{\partial x} \mathrm{d}x \, \mathrm{d}y, \qquad S_{ij}^{yy} = \int_{\Omega^e} \frac{\partial \hat{\phi}_i}{\partial y} \frac{\partial \hat{\phi}_j}{\partial y} \mathrm{d}x \, \mathrm{d}y \tag{26}$$

and $N_{1j}^{\alpha}, N_{2j}^{\alpha}, \ldots$, are given in Appendix B.

Note that the stiffness matrix evaluation requires the computation of the second derivatives of the (Hermite cubic) interpolation used for the transverse deflection. In the present study, we use the Lagrange linear interpolation of the geometry (i.e., the same as that used for variables u, v, ϕ_1, and ϕ_2):

$$x = \sum_{j=1}^{n} x_j \psi_j, \qquad y = \sum_{j=1}^{n} y_j \psi_j \tag{27}$$

Thus, isoparametric elements are used for (u,v,ϕ_1,ϕ_2) and the subparametric formulation is used for w. We must develop the transformation equations to numerically evaluate the stiffness coefficients. These relations can be found in Ref. 24, pp. 435–436.

The finite element based on the first-order shear deformation theory (set $\alpha = 0$ in the above equations) has five degrees of freedom per node when the same degree of interpolation is used for all variables. This element is known to have "locking" problems due to inconsistencies in the modeling of transverse shear energy, and it has been a subject of many papers in the literature.[4,25-30] The locking is avoided in most cases by using reduced integration in the numerical evaluation of element stiffnesses coming from the transverse shear energy. A consistent interpolation of the field variabiles, i.e., different or selective interpolations for w and (ϕ_1, ϕ_2) should prove to be more accurate in problems with variable coefficients.[31] The displacement finite-element model based on the third-order theory is less sensitive to locking.[32,33]

V. Numerical Results

The Lévy-type solution procedure and finite-element model developed in the previous sections are used to evaluate the natural frequencies and critical buckling loads of antisymmetric, cross-ply, rectangular laminates. The following material properties are used in the analysis: $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, and $\nu_{12} = 0.25$. All layers are assumed to have the same thickness and orthotropic material properties in the material principal axes. The shear correction factors $(K_{44}^2 = K_{55}^2)$ for FSDT are taken to be 5/6.

In the finite-element analysis, a mesh of 2 × 2 quadratic elements is used for the FSDT and a 4 × 4 mesh of 4-node elements is used for HSDT and CPT. Half-plate models are used in the free-vibration case and full plate models are used in the stability analysis.

The effect of orthotropy and number of layers on the nondimensional fundamental frequency of simply supported square laminates (0 deg/90 deg/0 deg/...) is investigated, and the numerical results obtained using various plate theories are compared in Table 1. The results obtained by Noor[34,35] are based on the three-dimensional elasticity theory. The fundamental frequencies increase with increasing orthotropy (E_1/E_2) as well as number of layers. Similar results for critical buckling loads are presented in Table 2.

The effect of including transverse shear strains and boundary conditions on the fundamental frequencies of two-layer and ten-layer antisymmetric cross-ply laminates are examined, and the results are presented in Table 3. In all cases, the classical plate theory overpredicts the frequencies compared to the shear-deformation theories. Frequencies predicted by the two shear-deformation theories are very close to each other. Similar results for buckling analysis are presented in Table 4.

The effect of laminate aspect ratio on fundamental frequencies are investigated for various boundary conditions and number of layers (see Table 5). The natural frequencies increase with increasing aspect ratio and number of layers.

In all cases, the finite-element solutions are in good agreement with the analytical solutions. Of course, the finite-element model is applicable to a more general case of lamination schemes, geometry, and boundary conditions. As an example, the model is applied to the natural vibration of cantilever laminates, and the results are presented in Table 6 for various aspect ratios and side-to-thickness ratios.

VI. Summary

Exact analytical solutions and finite-element numerical solutions are developed for natural vibration and buckling of antisymmetric, cross-ply, rectangular laminates under various boundary conditions on parallel edges when the other two

Table 3 Effect of side-to-thickness ratio on the dimensionless fundamental frequencies of antisymmetric cross-ply square plates including various boundary conditions $(b/a = 1)$: $\bar{\omega} = (\omega b^2/h)(\rho/E_2)^{1/2}$

Number of layers	b/h	Theory	Type of solution	SSFF[a]	SSFS[a]	SSFC[a]	SSSS[a]	SSSC[a]	SSCC[a]
2	5	HSDT	Exact	6.128	6.387	6.836	9.087	10.393	11.890
			FEM	6.172	6.192	6.648	9.103	10.582	12.053
2	5	FSDT	Exact	5.952	6.213	6.638	8.833	9.822	10.897
			FEM	5.955	6.219	6.646	8.837	9.899	10.906
2	5	CPT	Exact	7.124	7.450	8.041	10.721	13.627	17.741
			FEM	7.150	7.279	7.802	11.192	15.357	18.694
2	10	HSDT	Exact	6.943	7.277	7.810	10.568	12.870	15.709
			FEM	6.915	7.134	7.680	10.594	13.180	15.914
2	10	FSDT	Exact	6.881	7.215	7.741	10.473	12.610	15.152
			FEM	6.886	7.222	7.714	10.480	12.791	15.181
2	10	CPT	Exact	7.267	7.636	8.228	11.154	14.223	18.543
			FEM	7.262	7.345	7.821	11.383	14.828	19.053
10	5	HSDT	Exact	8.155	8.288	8.966	11.673	12.514	13.568
			FEM	7.989	7.998	8.694	11.664	12.633	13.710
10	5	FSDT	Exact	8.139	8.264	8.919	11.644	12.197	12.923
			FEM	8.143	8.270	8.925	11.647	12.239	12.928
10	5	CPT	Exact	11.459	11.815	13.618	12.167	23.348	30.855
			FEM	12.156	11.260	11.980	18.624	24.118	31.855
10	10	HSDT	Exact	10.893	11.074	11.863	15.771	18.175	20.831
			FEM	10.906	11.088	11.788	15.787	18.214	20.493
10	10	FSDT	Exact	10.900	11.079	11.862	15.779	18.044	20.471
			FEM	10.906	11.088	11.788	15.787	18.214	20.493
10	10	CPT	Exact	12.680	12.906	13.779	18.492	23.971	31.709
			FEM	12.419	11.283	11.983	18.637	23.991	31.912

[a]See Eqs. (20–22) for the explanation of S, F, and C. For example, SSFC means: the rectangular laminate is simply-supported (SS) at $y = 0$ and $y = b$, free (F) at $x = a/2$, and clamped (C) at $x = -a/2$.

Table 4 Effect of side-to-thickness ratio on the dimensionless critical buckling loads of antisymmetric cross-ply square plates including various boundary conditions: $\bar{N}_2 = \bar{N}_2 b^2/(E_2 h^3)$, $\bar{N}_1 = 0$

Number layers	b/h	Theory	Type of solution	SSFF	SSFS	SSFC	SSSS	SSSC	SSCC
2	5	HSDT	Exact	3.905	4.283	4.908	8.769	10.754	11.490
			FEM	3.979	4.375	5.022	8.985	11.241	12.318
2	5	FSDT	Exact	3.682	4.054	4.632	8.277	9.309	9.757
			FEM	3.719	4.094	4.667	8.328	9.650	9.949
2	5	CPT	Exact	5.425	6.003	6.968	12.957	21.116	31.280
			FEM	5.616	6.292	7.203	14.520	23.869	37.106
2	10	HSDT	Exact	4.940	5.442	6.274	11.562	17.133	21.464
			FEM	5.090	5.621	6.487	12.011	18.257	24.262
2	10	FSDT	Exact	4.851	5.351	6.166	11.353	16.437	20.067
			FEM	4.916	5.420	6.234	11.485	18.338	21.916
2	10	CPT	Exact	5.425	6.003	6.968	12.957	21.116	31.280
			FEM	5.616	6.292	7.203	14.520	23.869	37.106
10	5	HSDT	Exact	6.780	7.050	8.221	12.109	12.607	13.254
			FEM	6.802	7.089	8.278	12.224	12.800	13.659
10	5	FSDT	Exact	6.750	7.020	8.143	11.494	11.495	11.628
			FEM	6.791	7.064	8.174	11.172	11.181	11.216
10	5	CPT	Exact	16.426	17.023	19.389	35.232	59.288	89.770
			FEM	16.457	17.141	19.422	36.384	60.406	90.833
10	10	HSDT	Exact	12.077	12.506	14.351	25.423	32.885	35.376
			FEM	12.248	12.699	14.568	25.828	33.662	36.657
10	10	FSDT	Exact	12.092	12.524	14.358	25.450	32.614	34.837
			FEM	12.226	12.661	14.480	25.647	33.970	36.129
10	10	CPT	Exact	16.426	17.023	19.389	35.232	59.288	89.770
			FEM	16.457	17.141	19.422	36.384	60.406	90.833

Table 5 Effect of the aspect ratio on the dimensionless fundamental frequencies of antisymmetric cross-ply square plates including various boundary conditions: $b/h = 10$, $\bar{\omega} = (\omega b^2/h)(\rho/E_2)^{1/2}$

Number of layers	b/a	Theory	Type of solution	SSFF	SSFS	SSFC	SSSS	SSSC	SSCC
2	2	HSDT	Exact	6.943	8.171	12.632	26.301	33.387	40.925
			FEM	6.954	8.135	12.674	26.288	34.312	41.602
2	2	FSDT	Exact	6.881	8.109	12.395	25.608	31.111	36.723
			FEM	6.886	8.120	12.392	25.620	31.512	36.764
2	2	CPT	Exact	7.267	8.677	13.915	30.468	45.554	64.832
			FEM	7.318	8.702	13.959	31.312	48.880	67.652
2	3	HSDT	Exact	6.943	9.387	21.675	48.704	58.836	70.380
			FEM	6.963	9.434	21.836	48.582	60.621	71.508
2	3	FSDT	Exact	6.881	9.319	20.779	46.360	52.295	59.293
			FEM	6.886	9.333	20.786	46.377	52.770	59.328
2	3	CPT	Exact	7.267	10.153	25.769	63.325	96.451	137.71
			FEM	7.326	10.370	25.988	65.985	105.71	145.87
10	2	HSDT	Exact	10.893	11.577	18.110	34.747	39.223	44.598
			FEM	10.882	11.394	18.067	34.652	39.998	45.274
10	2	FSDT	Exact	10.900	11.577	17.930	34.682	37.689	41.520
			FEM	10.906	11.589	17.916	34.692	37.964	41.538
10	2	CPT	Exact	12.680	13.569	22.876	52.292	79.371	113.80
			FEM	12.682	12.977	22.426	52.924	81.366	115.68
10	3	HSDT	Exact	10.893	12.324	28.658	57.523	63.596	71.800
			FEM	10.899	12.272	28.886	57.324	65.169	72.864
10	3	FSDT	Exact	10.900	12.313	27.764	56.967	58.987	63.042
			FEM	10.906	12.326	27.764	56.976	59.310	63.054
10	3	CPT	Exact	12.680	14.606	43.616	111.58	159.65	159.95
			FEM	12.720	14.536	43.554	114.29	161.11	160.96

Table 6 Fundamental frequencies of a cantilever laminate (0 deg/90 deg) as predicted by various theories: $\bar{\omega} = (\omega b^2/h)(\rho/E_2)^{1/2}$

	CPT		FSDT		HSDT	
b/a	$b/h = 100$	$b/h = 10$	$b/h = 100$	$b/h = 10$	$b/h = 100$	$b/h = 10$
1	2.6285	2.6250	2.6103	2.5334	2.6378	2.5610
2	10.5138	10.4588	10.4318	9.3501	10.5385	9.5988
3	23.6548	23.3775	23.4354	18.8491	23.6666	19.8325

edges are simply supported. The effects of aspect ratio, boundary conditions, side-to-thickness ratio, and material orthotropy on fundamental frequencies and critical buckling loads are investigated. The finite-element solutions are found to be in good agreement with the analytical solutions. In general, the classical plate theory overpredicts natural frequencies and buckling loads, and the difference increases with increasing side-to-thickness ratio. The analytical solutions and numerical results presented here can serve to validate other methods and finite-element models.

Appendix A: The Matrix [A] Coefficients

HSDT:

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & 0 & C_4 & C_5 & 0 & 0 & C_6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_7 & C_8 & 0 & C_9 & 0 & C_{10} & 0 & 0 & C_{11} & C_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_{13} & C_{14} & 0 & C_{15} & 0 & C_{16} & 0 & 0 & C_{17} & C_{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{19} & 0 & 0 & C_{20} & 0 & C_{21} & 0 & C_{22} & C_{23} & 0 & 0 & C_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{25} & C_{26} & 0 & C_{27} & 0 & C_{28} & 0 & 0 & C_{29} & C_{30} & 0 \end{bmatrix} \tag{A1}$$

$C_1 = (e_7 e_{30} - e_3 e_{34})/e_0, \qquad C_2 = (e_2 e_{30} - e_3 e_{29})/e_0$

$C_3 = (e_6 e_{30} - e_3 e_{33})/e_0, \qquad C_4 = (e_5 e_{30} - e_3 e_{32})/e_0$

$C_5 = (e_8 e_{30} - e_3 e_{35})/e_0, \qquad C_6 = (e_4 e_{30} - e_3 e_{31})/e_0$

$C_7 = (e_9 e_{39} - e_{12} e_{36})/C_0, \qquad C_8 = (e_{14} e_{39} - e_{12} e_{41})/C_0$

$C_9 = (e_{16} e_{39} - e_{12} e_{43})/C_0, \qquad C_{10} = (e_{13} e_{39} - e_{12} e_{40})/C_0$

$C_{11} = (e_{11} e_{39} - e_{12} e_{38})/C_0, \qquad C_{12} = (e_{15} e_{39} - e_{12} e_{42})/C_0$

$C_{19} = (e_1 e_{34} - e_7 e_{28})/e_0, \qquad C_{20} = (e_1 e_{29} - e_2 e_{28})/e_0$

$C_{21} = (e_1 e_{33} - e_6 e_{28})/e_0, \qquad C_{22} = (e_1 e_{32} - e_5 e_{28})/e_0$

$C_{23} = (e_1 e_{35} - e_8 e_{28})/e_0, \qquad C_{24} = (e_1 e_{31} - e_4 e_{28})/e_0$

$C_{25} = (e_{10} e_{36} - e_9 e_{37})/C_0, \qquad C_{26} = (e_{10} e_{41} - e_{14} e_{37})/C_0$

$C_{27} = (e_{10} e_{43} - e_{16} e_{37})/C_0, \qquad C_{28} = (e_{10} e_{40} - e_{13} e_{37})/C_0$

$C_{29} = (e_{10} e_{38} - e_{11} e_{37})/C_0, \qquad C_{30} = (e_{10} e_{42} - e_{15} e_{37})/C_0$

$C_{13} = a_0(C_1 e_{21} + C_7 a_1 + C_{25} a_2 + C_{19} e_{23} + e_{26})$

$C_{14} = a_0(C_8 a_1 + C_{26} a_2 + e_{27}), \qquad C_{15} = a_0(C_9 a_1 + C_{27} a_2 + e_{20})$

$C_{16} = a_0(e_{18} + C_3 e_{21} + C_{21} e_{23} + C_{10} a_1 + C_{28} a_2)$

$C_{17} = a_0(e_{17} + C_5 e_{21} + C_{23} e_{23} + C_{11} a_1 + C_{29} a_2)$

$$C_{18} = a_0(e_{19} + C_{12} a_1 + C_{30} a_2) \tag{A2}$$

and

$e_0 = e_3 e_{28} - e_1 e_{30}$

$C_0 = e_{12} e_{37} - e_{10} e_{39}, \qquad a_0 = -1/(C_4 e_{21} + C_{22} e_{23} + e_{25})$

$a_1 = C_2 e_{21} + C_{20} e_{23} + e_{22}, \qquad a_2 = C_6 e_{21} + C_{24} e_{23} + e_{24}$

$e_1 = A_{11}, \qquad e_2 = -\beta(A_{12} + A_{66}), \qquad e_3 = B_{11} - \frac{4}{3h^2} E_{11}$

$e_4 = \beta\left[\frac{4}{3h^2}(E_{12} + E_{66}) - B_{12} - B_{66}\right], \qquad e_5 = -\frac{4}{3h^2} E_{11}$

$e_6 = \frac{4}{3h^2}[\beta^2(E_{12} + 2E_{66}) - I_4 \omega_m^2], \qquad e_7 = -\beta^2 A_{66} + I_1 \omega_m^2$

$e_8 = \beta^2\left(\frac{4}{3h^2} E_{66} - B_{66}\right) + \bar{I}_2 \omega_m^2, \qquad e_9 = -e_2,$

$e_{10} = A_{66}, \qquad e_{11} = -e_4, \qquad e_{12} = B_{66} - \frac{4}{3h^2} E_{66},$

$e_{13} = -\frac{4}{3h^2}\beta(E_{12} + 2E_{66}), \qquad e_{14} = -\beta^2 A_{22} + I_1 \omega_m^2$

$e_{15} = \beta^2\left(\frac{4}{3h^2} E_{22} - B_{22}\right) + \bar{I}_2 \omega_m^2, \quad e_{16} = \frac{4}{3h^2}(\beta^3 E_{22} - \beta I_4 \omega_m^2)$

$$e_{17} = A_{55} - \frac{4}{h^2} D_{55} - \frac{4}{h^2}\left(D_{55} - \frac{4}{h^2} F_{55}\right) + \frac{4}{3h^2}\beta^2\left[\frac{4}{3h^2}(H_{12} + 2H_{66}) - (F_{12} + 2F_{66})\right] + \frac{4}{3h^2}\bar{I}_5 \omega_m^2$$

$$e_{18} = A_{55} - \frac{4}{h^2} D_{55} - \frac{4}{h^2}\left(D_{55} - \frac{4}{h^2} F_{55}\right) + \left(\frac{4}{3h^2}\right)^2 \beta^2[2H_{12} + 4H_{66}] - \bar{N}_1 - \left(\frac{4}{3h^2}\right)^2 I_7 \omega_m^2$$

$$e_{19} = -\beta\left[A_{44} - \frac{4}{h^2} D_{44} - \frac{4}{h^2}\left(D_{44} - \frac{4}{h^2} F_{44}\right)\right] + \frac{4}{3h^2}\beta^3\left(F_{22} - \frac{4}{3h^2} H_{22}\right) - \frac{4}{3h^2}\beta \bar{I}_5 \omega_m^2$$

$$e_{20} = -\beta^2\left[A_{44} - \frac{4}{h^2}D_{44} - \frac{4}{h^2}\left(D_{44} - \frac{4}{h^2}F_{44}\right)\right] - \left(\frac{4}{3h^2}\right)^2\beta^4 H_{22} + \beta^2\bar{N}_2 + I_1\omega_m^2 + \left(\frac{4}{3h^2}\right)^2\beta^2 I_7\omega_m^2$$

$$e_{21} = -e_5, \quad e_{22} = e_{13}, \quad e_{23} = \frac{4}{3h^2}\left(F_{11} - \frac{4}{3h^2}H_{11}\right)$$

$$e_{24} = \frac{4}{3h^2}\beta\left[\frac{4}{3h^2}(2H_{66} + H_{12}) - (F_{12} + 2F_{66})\right]$$

$$e_{25} = -\left(\frac{4}{3h^3}\right)^2 H_{11}$$

$$e_{26} = -e_6, \quad e_{27} = e_{16}, \quad e_{28} = e_3, \quad e_{29} = e_4$$

$$e_{30} = D_{11} - \frac{8}{3h^2}F_{11} + \left(\frac{4}{3h^2}\right)^2 H_{11}$$

$$e_{31} = \beta\left[\frac{8}{3h^2}(F_{12} + F_{66}) - \left(\frac{4}{3h^2}\right)^2(H_{12} + H_{66}) - D_{12} - D_{66}\right]$$

$$e_{32} = -e_{23}, \quad e_{33} = -e_{17}, \quad e_{34} = e_8$$

$$e_{35} = \frac{4}{h^2}\left(D_{55} - \frac{4}{h^2}F_{55}\right) - \left(A_{55} - \frac{4}{h^2}D_{55}\right) + \beta^2\left[\frac{8}{3h^2}F_{66} - D_{66} - \left(\frac{4}{3h^2}\right)^2 H_{66}\right] + \bar{I}_3\omega_m^2$$

$$e_{36} = -e_4, \quad e_{37} = e_{12}, \quad e_{38} = -e_{31},$$

$$e_{39} = D_{66} - \frac{8}{3h^2}F_{66} + \left(\frac{4}{3h^2}\right)^2 H_{66}$$

$$e_{40} = e_{24}, \quad e_{41} = e_{15}$$

$$e_{42} = \frac{4}{h^2}\left(D_{44} - \frac{4}{h^2}F_{44}\right) - \left(A_{44} - \frac{4}{h^2}D_{44}\right) + \beta^2\left[\frac{8}{3h^2}F_{22} - D_{22} - \left(\frac{4}{3h^2}\right)^2 H_{22}\right] + \bar{I}_3\omega_m^2$$

$$e_{43} = e_{19} \tag{A3}$$

FSDT:

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & C_4 & 0 & 0 & C_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & C_7 & 0 & C_8 & 0 & 0 & C_9 & C_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{11} & 0 & 0 & C_{12} & C_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{14} & 0 & 0 & C_{15} & 0 & C_{16} & C_{17} & 0 & 0 & C_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_{19} & C_{20} & 0 & C_{21} & 0 & 0 & C_{22} & C_{23} & 0 \end{bmatrix} \tag{A4}$$

$$C_1 = (e_3e_{21} - e_5e_{19})/e_0, \quad C_2 = (e_3e_{18} - e_2e_{19})/e_0$$

$$C_3 = e_3e_{23}/e_0, \quad C_4 = (e_3e_{22} - e_6e_{19})/e_0$$

$$C_5 = (e_3e_{20} - e_4e_{19})/e_0, \quad C_6 = (e_8e_{27} - e_{10}e_{25})/C_0$$

$$C_7 = (e_{11}e_{27} - e_{10}e_{28})/C_0, \quad C_8 = -e_{10}e_{30}/C_0$$

$$C_9 = (e_9e_{27} - e_{10}e_{26})/C_0, \quad C_{10} = (e_{12}e_{27} - e_{10}e_{29})/C_0$$

$$C_{11} = -e_{15}/e_{13}$$

$$C_{12} = -e_{14}/e_{13}, \quad C_{13} = -e_{16}/e_{13}, \quad C_{14} = (e_5e_{17} - e_1e_{21})/e_0$$

$$C_{15} = (e_2e_{17} - e_1e_{18})/e_0, \quad C_{16} = -e_1e_{23}/e_0$$

$$C_{17} = (e_6e_{17} - e_1e_{22})/e_0, \quad C_{18} = (e_4e_{17} - e_1e_{20})/e_0$$

$$C_{19} = (e_7e_{25} - e_8e_{24})/C_0, \quad C_{20} = (e_7e_{28} - e_{11}e_{24})/C_0$$

$$C_{21} = e_7e_{30}/C_0, \quad C_{22} = (e_7e_{26} - e_9e_{24})/C_0$$

$$C_{23} = (e_7e_{29} - e_{12}e_{24})/C_0, \quad e_0 = e_1e_{19} - e_3e_{17}$$

$$C_0 = e_{10}e_{24} - e_7e_{27} \tag{A5}$$

and

$$e_1 = A_{11}, \quad e_2 = -\beta(A_{12} + A_{66}), \quad e_3 = B_{11}$$

$$e_4 = -\beta(B_{12} + B_{66}), \quad e_5 = -\beta^2A_{66} + I_1\omega_m^2$$

$$e_6 = -\beta^2B_{66} + I_2\omega_m^2, \quad e_7 = A_{66}, \quad e_8 = -e_2$$

$$e_9 = -e_4, \quad e_{10} = B_{66}, \quad e_{11} = -\beta^2A_{22} + I_1\omega_m^2$$

$$e_{12} = -\beta^2B_{22} + I_2\omega_m^2, \quad e_{13} = K_{55}^2A_{55} - \bar{N}_1$$

$$e_{14} = K_{55}^2A_{55}, \quad e_{15} = -\beta^2K_{44}^2A_{44} + I_1\omega_m^2 + \beta^2\bar{N}_2$$

$$e_{16} = -\beta K_{44}^2A_{44}, \quad e_{17} = e_3, \quad e_{18} = e_4$$

$$e_{19} = D_{11}, \quad e_{20} = -\beta(D_{11} + D_{66})$$

$$e_{21} = -\beta^2B_{66} + I_2\omega_m^2, \quad e_{22} = -\beta^2D_{66} - K_{55}^2A_{55} + I_3\omega_m^2$$

$$e_{23} = -e_{14}, \quad e_{24} = e_{10}, \quad e_{25} = -e_4, \quad e_{26} = -e_{20}$$

$$e_{27} = D_{66}, \quad e_{28} = -\beta^2B_{22} + I_2\omega_m^2$$

$$e_{29} = -\beta^2D_{22} - K_{44}^2A_{44} + I_3\omega_m^2, \quad e_{30} = e_{16} \tag{A6}$$

CPT:

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & C_2 & 0 & C_3 & 0 & C_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & C_5 & C_6 & 0 & C_7 & 0 & C_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & C_9 & C_{10} & 0 & C_{11} & 0 & C_{12} & 0 \end{bmatrix} \tag{A7}$$

$$C_1 = -e_2/e_1, \quad C_2 = -e_3/e_1, \quad C_3 = -e_5/e_1$$

$$C_4 = -e_4/e_1, \quad C_5 = -e_6/e_7$$

$$C_6 = -e_8/e_7, \quad C_7 = -e_{10}/e_7, \quad C_8 = -e_9/e_7$$

$$C_9 = -e_{21}/e_{18}, \quad C_{10} = -e_{22}/e_{18}$$

$$C_{11} = -e_{20}/e_{18}, \quad C_{12} = -e_{19}/e_{18} \tag{A8}$$

and

$$e_1 = A_{11}, \quad e_2 = -\beta^2A_{66} + I_1\omega_m^2, \quad e_3 = -\beta(A_{12} + A_{66})$$

$$e_4 = -B_{11}, \quad e_5 = \beta^2(B_{12} + 2B_{66}) - I_2\omega_m^2$$

$$e_6 = -e_3, \quad e_7 = A_{66}, \quad e_8 = -\beta^2 A_{22} + I_1\omega_m^2$$
$$e_9 = -\beta(B_{12} + 2B_{66}), \quad e_{10} = \beta^3 B_{22} - \beta I_2 \omega_m^2, \quad e_{11} = D_{11}$$
$$e_{12} = -2\beta^2(D_{12} + 2D_{66}) + I_3\omega_m^2 + \bar{N}_1$$
$$e_{13} = \beta^4 D_{22} - I_1\omega_m^2 - \beta^2 I_3 \omega_m^2 - \beta^2 \bar{N}_2, \quad e_{14} = e_4, \quad e_{15} = e_5$$
$$e_{16} = -e_9, \quad e_{17} = -e_{10}, \quad e_{18} = e_{11} - e_4 e_{14}/e_1$$
$$e_{19} = e_{12} - e_5 e_{14}/e_1 - e_9 e_{16}/e_7 + e_3 e_9 e_{14}/(e_1 e_7)$$
$$e_{20} = e_{13} - e_{10} e_{16}/e_7 + e_3 e_{10} e_{14}/(e_1 e_7)$$
$$e_{21} = e_{15} - e_2 e_{14}/e_1 - e_6 e_{16}/e_7 + e_3 e_6 e_{14}/(e_1 e_7)$$
$$e_{22} = e_{17} - e_8 e_{16}/e_7 + e_3 e_8 e_{14}/(e_1 e_7) \tag{A9}$$

Appendix B: Finite-Element Stiffness Coefficients

$$N^1_{1j} = A_{11}\frac{\partial\psi_j}{\partial x} + A_{16}\frac{\partial\psi_j}{\partial y}, \qquad N^2_{1j} = A_{16}\frac{\partial\psi_j}{\partial x} + A_{12}\frac{\partial\psi_j}{\partial y}$$
$$N^3_{1j} = -c_2\left(E_{11}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2E_{16}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + E_{12}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$N^4_{1j} = \hat{B}_{11}\frac{\partial\psi_j}{\partial x} + \hat{B}_{16}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{1j} = \hat{B}_{12}\frac{\partial\psi_j}{\partial y} + \hat{B}_{16}\frac{\partial\psi_j}{\partial x}$$
$$N^1_{6j} = A_{16}\frac{\partial\psi_j}{\partial x} + A_{66}\frac{\partial\psi_j}{\partial y}, \qquad N^2_{6j} = A_{66}\frac{\partial\psi_j}{\partial x} + A_{26}\frac{\partial\psi_j}{\partial y}$$
$$N^3_{6j} = -c_2\left(E_{16}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2E_{66}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + E_{26}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$N^4_{6j} = \hat{B}_{16}\frac{\partial\psi_j}{\partial x} + \hat{B}_{66}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{6j} = \hat{B}_{26}\frac{\partial\psi_j}{\partial y} + \hat{B}_{66}\frac{\partial\psi_j}{\partial x}$$
$$N^1_{2j} = A_{12}\frac{\partial\psi_j}{\partial x} + A_{26}\frac{\partial\psi_j}{\partial y}, \qquad N^2_{2j} = A_{26}\frac{\partial\psi_j}{\partial x} + A_{22}\frac{\partial\psi_j}{\partial y}$$
$$N^3_{2j} = -c_2\left(E_{12}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2E_{26}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + E_{22}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$N^4_{2j} = \hat{B}_{12}\frac{\partial\psi_j}{\partial x} + \hat{B}_{26}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{2j} = \hat{B}_{22}\frac{\partial\psi_j}{\partial y} + \hat{B}_{26}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^1_{1j} = \hat{B}_{11}\frac{\partial\psi_j}{\partial x} + \hat{B}_{16}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^2_{2j} = \hat{B}_{16}\frac{\partial\psi_j}{\partial x} + \hat{B}_{12}\frac{\partial\psi_j}{\partial y}$$
$$\hat{M}^3_{1j} = -c_2\left(\hat{F}_{11}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2\hat{F}_{16}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + \hat{F}_{12}\frac{\partial^2\hat{\phi}_j}{\partial x^2}\right)$$
$$\hat{M}^4_{1j} = \bar{D}_{11}\frac{\partial\psi_j}{\partial x} + \bar{D}_{16}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{1j} = \bar{D}_{12}\frac{\partial\psi_j}{\partial y} + \bar{D}_{16}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^1_{6j} = \hat{B}_{16}\frac{\partial\psi_j}{\partial x} + \hat{B}_{66}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^2_{6j} = \hat{B}_{66}\frac{\partial\psi_j}{\partial x} + B_{26}\frac{\partial\psi_j}{\partial y}$$
$$\hat{M}^3_{6j} = -c_2\left(\hat{F}_{16}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2\hat{F}_{66}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + \hat{F}_{26}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$\hat{M}^4_{6j} = \bar{D}_{16}\frac{\partial\psi_j}{\partial x} + \bar{D}_{66}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{6j} = \bar{D}_{26}\frac{\partial\psi_j}{\partial y} + \bar{D}_{66}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^1_{2j} = \hat{B}_{12}\frac{\partial\psi_j}{\partial x} + \hat{B}_{26}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^2_{2j} = \hat{B}_{26}\frac{\partial\psi_j}{\partial x} + \hat{B}_{22}\frac{\partial\psi_j}{\partial y}$$
$$\hat{M}^3_{2j} = -c_2\left(F_{12}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2F_{26}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + F_{22}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$\hat{M}^4_{2j} = \bar{D}_{12}\frac{\partial\psi_j}{\partial x} + \bar{D}_{26}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{2j} = \bar{D}_{22}\frac{\partial\psi_j}{\partial y} + \bar{D}_{26}\frac{\partial\psi_j}{\partial x}$$
$$P^1_{1j} = E_{11}\frac{\partial\psi_j}{\partial x} + E_{16}\frac{\partial\psi_j}{\partial y}, \qquad P^2_{1j} = E_{12}\frac{\partial\psi_j}{\partial y} + E_{16}\frac{\partial\psi_j}{\partial x}$$
$$P^3_{1j} = -c_2\left(H_{11}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2H_{16}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + H_{12}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$P^4_{1j} = \hat{F}_{11}\frac{\partial\psi_j}{\partial x} + \hat{F}_{16}\frac{\partial\psi_j}{\partial y}, \qquad P^5_{1j} = \hat{F}_{12}\frac{\partial\psi_j}{\partial x} + \hat{F}_{16}\frac{\partial\psi_j}{\partial x}$$
$$P^1_{2j} = E_{12}\frac{\partial\psi_j}{\partial x} + E_{26}\frac{\partial\psi_j}{\partial y}, \qquad P^2_{2j} = E_{22}\frac{\partial\psi_j}{\partial y} + E_{26}\frac{\partial\psi_j}{\partial x}$$
$$P^3_{2j} = -c_2\left(H_{21}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2H_{26}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + H_{22}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$P^4_{2j} = \hat{F}_{12}\frac{\partial\psi_j}{\partial x} + \hat{F}_{26}\frac{\partial\psi_j}{\partial y}, \qquad P^5_{2j} = \hat{F}_{22}\frac{\partial\psi_j}{\partial y} + \hat{F}_{26}\frac{\partial\psi_j}{\partial x}$$
$$P^1_{6j} = E_{16}\frac{\partial\psi_j}{\partial x} + E_{66}\frac{\partial\psi_j}{\partial y}, \qquad P^2_{6j} = E_{26}\frac{\partial\psi_j}{\partial y} + E_{66}\frac{\partial\psi_j}{\partial x}$$
$$P^3_{6j} = -c_2\left(H_{16}\frac{\partial^2\hat{\phi}_j}{\partial x^2} + 2H_{66}\frac{\partial^2\hat{\phi}_j}{\partial x\,\partial y} + H_{26}\frac{\partial^2\hat{\phi}_j}{\partial y^2}\right)$$
$$P^4_{6j} = \hat{F}_{16}\frac{\partial\psi_j}{\partial x} + \hat{F}_{66}\frac{\partial\psi_j}{\partial y}, \qquad P^5_{6j} = \hat{F}_{26}\frac{\partial\psi_j}{\partial y} + \hat{F}_{66}\frac{\partial\psi_j}{\partial x}$$
$$\bar{D}_{ij} = \hat{D}_{ij} - c_2\hat{F}_{ij}, \qquad i,j = 1,2,6$$
$$N^4_{1j} = \hat{B}_{11}\frac{\partial\psi_j}{\partial x} + \hat{B}_{16}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{1j} = \hat{B}_{12}\frac{\partial\psi_j}{\partial y} + \hat{B}_{16}\frac{\partial\psi_j}{\partial x}$$
$$N^4_{2j} = \hat{B}_{12}\frac{\partial\psi_j}{\partial x} + \hat{B}_{26}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{2j} = \hat{B}_{22}\frac{\partial\psi_j}{\partial y} + \hat{B}_{26}\frac{\partial\psi_j}{\partial x}$$
$$N^4_{6j} = \hat{B}_{16}\frac{\partial\psi_j}{\partial x} + \hat{B}_{66}\frac{\partial\psi_j}{\partial y}, \qquad N^5_{6j} = \hat{B}_{26}\frac{\partial\psi_j}{\partial y} + \hat{B}_{66}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^4_{1j} = \bar{D}_{11}\frac{\partial\psi_j}{\partial x} + \bar{D}_{16}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{1j} = \bar{D}_{12}\frac{\partial\psi_j}{\partial y} + \bar{D}_{16}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^4_{2j} = \bar{D}_{21}\frac{\partial\psi_j}{\partial x} + \bar{D}_{26}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{2j} = \bar{D}_{22}\frac{\partial\psi_j}{\partial y} + \bar{D}_{26}\frac{\partial\psi_j}{\partial x}$$
$$\hat{M}^4_{6j} = \bar{D}_{16}\frac{\partial\psi_j}{\partial x} + \bar{D}_{66}\frac{\partial\psi_j}{\partial y}, \qquad \hat{M}^5_{6j} = \bar{D}_{26}\frac{\partial\psi_j}{\partial y} + \bar{D}_{66}\frac{\partial\psi_j}{\partial x}$$
$$\hat{Q}^3_{1j} = \bar{A}_{45}\frac{\partial\hat{\phi}_j}{\partial y} + \bar{A}_{55}\frac{\partial\hat{\phi}_j}{\partial x}, \qquad \hat{Q}^3_{2j} = \bar{A}_{44}\frac{\partial\hat{\phi}_j}{\partial y} + \bar{A}_{45}\frac{\partial\hat{\phi}_j}{\partial x}$$
$$\hat{Q}^4_{1j} = \bar{A}_{55}\psi_j, \qquad \hat{Q}^5_{1j} = \bar{A}_{45}\psi_j, \qquad \hat{Q}^4_{2j} = \bar{A}_{45}\psi_j$$
$$\hat{Q}^5_{2j} = \bar{A}_{44}\psi_j \qquad \bar{A}_{ij} = \hat{A}_{ij} - c_1\hat{D}_{ij}, \qquad i,j = 4,5$$

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